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SOME TOPICS IN PG-GEOMETRY

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Our interest is to construct "good"cohomology theories on complex varieties with singularity. PG-geometry will be the first step for it. In this note we shall formulate relative cohomology theory, cohomology theory with supports and homology theory, with polynomial growth condition and discuss their elementary properties and relations between them. Our discussions are all elementary so we omit the proofs.

Let $(X, \mathcal{O}_X)$ be a reduced complex space and $g:X \to [1,+\infty[$ a function. A holomorphic function f on an open set $U \subset X$ is called a PG-holomorphic function with respect to g (or simply a PG-holomorphic function) if there exists $\alpha=(\alpha_1,\alpha_2) \in [1,+\infty[\times [1,+\infty[$ such that

$$|f(x)| \leq \alpha_1 g^{\alpha_2} \quad \text{for all} \quad x \in U.$$

We define a linear subspace ${}_{PG}\Gamma(U, \mathcal{O}_X)$ of $\mathcal{O}_X(U)$ as the set of all PG-holomorphic functions on U.

We shall keep in mind following two situations:

(I) X is a smooth affine variety and $g(z):=1+|z|$.

(II) Let G be an open set in $\mathbb{C}^N$. then $X \subset G$ is an analytic variety, $D:=V(h)$ a locus of $h \in \mathcal{O}(G)$ in X such that $U:=X \setminus D$ is smooth, and $g(z) :=|h(z)|^{-1}$.

For simplicity we state only in the case (I).

§3. Relative PG-cohomology by PG-coverings.

3.1. Relative PG-coverings.

(3.1.0) Let X be a smooth affine variety in $\mathbb{C}^N$, $Z \subset X$ a closed subvariety and $Y := X \setminus Z$. Let $\hat{A}_\sigma(X)$ be a PG-covering of X, i.e.,

$$\hat{A}_\sigma(X) := \{ \hat{\Delta}_\sigma(x) \}_{x \in X} \qquad \text{for } \sigma = (\sigma_1, \sigma_2) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$$

where $\hat{\Delta}_\sigma(x) := X \cap \{ y \mid \| y - x \| < (\sigma_1 (1+|z|)^{\sigma_2})^{-1} \}$.

(3.1.1) It is called that $(\hat{A}_\sigma(X), \hat{A}_\sigma(Y))$ is a relative PG-covering of (X, Y) if $\hat{A}_\sigma(X) = (U_i)_{i \in I}$ is a PG-covering of X and $\hat{A}_\sigma(Y) = (U_i)_{i \in I'}$ is a subcovering of Y with $I' \subset I$.

(3.1.2) Let $(\hat{A}_\sigma(X), \hat{A}_\sigma(Y))$ and $(\hat{A}_\rho(X), \hat{A}_\rho(Y))$ be two relative PG-coverings of (X, Y). We define

$$(\hat{A}_\sigma(X), \hat{A}_\sigma(Y)) < (\hat{A}_\rho(X), \hat{A}_\rho(Y))$$

as $\sigma < \rho$, i.e., $\sigma_1 < \rho_1$ and $\sigma_2 < \rho_2$, in other words,

a) for each $\beta \in J$ there exists $\alpha \in I$ such that $V_\beta \subset U_\alpha$,

b) for each $\beta \in J'$ there exists $\alpha \in I'$ such that $V_\beta \subset U_\alpha$,

where $\hat{A}_\rho(X) = (V_j)_{j \in J}$ and $\hat{A}_\rho(Y) = (V_j)_{j \in J'}$ with $J' \subset J$. Then we have a refinement map $\tau : J \to I$, $\beta \mapsto \tau(\beta) = \alpha$. It defines a simplicial map

$$\tau : (\hat{A}_\rho(X), \hat{A}_\rho(Y)) \longrightarrow (\hat{A}_\sigma(X), \hat{A}_\sigma(Y)).$$

3.2. Relative PG-cohomology with coefficients in structure sheaf $\mathcal{O}_X$

(3.2.1) Let $(\hat{A}_\sigma(X), \hat{A}_\sigma(Y)) = ((U_i)_{i \in I}, (U_i)_{i \in I'})$ be a relative PG-covering of (X, Y). For any integer $p \geq 0$,

$\varphi = (\varphi_{i_0 \cdots\cdots i_p}) \in \prod_{(i_0,\dots,i_p)\in I^{p+1}} \mathcal{O}(U_{i_0 \cdots i_p})$ is <u>a relative alternative p-cochain with coefficient in $\mathcal{O}$</u> if

a) $\varphi_{i_0 \cdots\cdots i_p} \in \mathcal{O}(U_{i_0 \cdots\cdots i_p}) = \mathcal{O}(U_{i_0} \cap \cdots\cdots \cap U_{i_p})$

b) $\varphi_{i_0 \cdots\cdots i_\alpha \cdots i_\beta \cdots\cdots i_p} = -\varphi_{i_0 \cdots\cdots i_\beta \cdots i_\alpha \cdots\cdots i_p}$

c) $\varphi_{i_0 \cdots\cdots i_p} = 0$ if $i_0, \cdots\cdots, i_p \in I'$.

The set of relative alternative p-cochains will be denoted by

$$C^p(\hat{A}_\sigma(X), \hat{A}_\sigma(Y) ; \mathcal{O}_X)$$

and is an abelian group.

(3.2.2) We define a coboundry operator

$$\delta : C^p(\hat{A}_\sigma(X), \hat{A}_\sigma(Y) ; \mathcal{O}_X) \longrightarrow C^{p+1}(\hat{A}_\sigma(X), \hat{A}_\sigma(Y) ; \mathcal{O}_X)$$

as follows. If $\varphi = (\varphi_{i_0 \cdots\cdots i_p}) \in C^p(\hat{A}_\sigma(X), \hat{A}_\sigma(Y) ; \mathcal{O}_X)$, define

$$(\delta\varphi)_{i_0 \cdots\cdots i_{p+1}} := \sum_{j=0}^{p+1} (-i)^j \operatorname{res}^{U_{i_0 \cdots \hat{i}_j \cdots i_{p+1}}}_{U_{i_0 \cdots\cdots i_{p+1}}} (\varphi_{i_0 \cdots\cdots \hat{i}_j \cdots\cdots i_{p+1}}).$$

It is clear that δ is a group homomorphism and that $\delta^2 = 0$. Thus we have a relative cochain complex

$$(C^\bullet(\hat{A}_\sigma(X), \hat{A}_\sigma(Y) ; \mathcal{O}_X), \delta).$$

(3.2.3) We define $_{PG}C^p(\hat{A}_\sigma(X), \hat{A}_\sigma(Y) ; \mathcal{O}_X) := \{\varphi \in C^p(\hat{A}_\sigma(X), \hat{A}_\sigma(Y) : \mathcal{O}_X) \mid \exists\, \alpha = (\alpha_1, \alpha_2) \in [1, +\infty[\times [1, +\infty[$ such that $|\varphi(x)| \leqslant \alpha_1 g^{\alpha_2}\}$.

Then

$$\begin{tikzcd}
C^p(\hat{A}_\sigma(X), \hat{A}_\sigma(Y) ; \mathcal{O}_X) \arrow[r, "\delta"] & C^{p+1}(\hat{A}_\sigma(X), \hat{A}_\sigma(Y) ; \mathcal{O}_X) \\
{}_{PG}C^p(\hat{A}_\sigma(X), \hat{A}_\sigma(Y) ; \mathcal{O}_X) \arrow[u, hook] \arrow[r, "\delta"] \arrow[ur, phantom, "\circlearrowleft"] & {}_{PG}C^{p+1}(\hat{A}_\sigma(X), \hat{A}_\sigma(Y) ; \mathcal{O}_X) \arrow[u, hook]
\end{tikzcd}$$

So we have a relative cochain complex $({}_{PG}C^{\bullet}(\hat{A}_{\sigma}(X),\hat{A}_{\sigma}(Y);\mathcal{O}_X),\delta)$ and define: ${}_{PG}H^p(\hat{A}_{\sigma}(X),\hat{A}_{\sigma}(Y);\mathcal{O}_X):=H^p({}_{PG}C^{\bullet}(\hat{A}_{\sigma}(X),\hat{A}_{\sigma}(Y);\mathcal{O}_X))$.

(3.2.4) Let $\tau:(\hat{A}_{\rho}(X),\hat{A}_{\rho}(Y))\longrightarrow(\hat{A}_{\sigma}(X),\hat{A}_{\sigma}(Y))$ be a simplicial map associated to a refinement map $\tau:J\to I$. Then τ induces a morphism $\tau^*:{}_{PG}C^p(\hat{A}_{\sigma}(X),\hat{A}_{\sigma}(Y);\mathcal{O}_X)\longrightarrow{}_{PG}C^p(\hat{A}_{\rho}(X),\hat{A}_{\rho}(Y);\mathcal{O}_X)$, defined by

$$(\tau^*\varphi)_{\beta_0\cdots\beta_p}=\varphi_{\tau(\beta_0)\cdots\tau(\beta_p)}\quad\text{on}\quad U_{\tau(\beta_0)\cdots\tau(\beta_p)},$$

and τ^* commutes with the coboundary operater δ. Consequently we have a homomorphism

$${}_{PG}H^p(\hat{A}_{\sigma}(X),\hat{A}_{\sigma}(Y);\mathcal{O}_X)\longrightarrow{}_{PG}H^p(\hat{A}_{\rho}(X),\hat{A}_{\rho}(Y);\mathcal{O}_X)$$

which is independent of the choice of the refinement map τ.

(3.2.5) Thus we can define :

$${}_{PG}H^p(X,Y;\mathcal{O}_X)={}_{PG}H^p(X\bmod Y;\mathcal{O}_X):=\varinjlim_{\sigma}{}_{PG}H^p(\hat{A}_{\sigma}(X),\hat{A}_{\sigma}(Y);\mathcal{O}_X)$$

called <u>p-th relative PG-cohomology group with coefficient in $\mathcal{O}_X$</u>.

3.3. Long exact sequence.

<u>PROPOSITION (3.3.1)</u> Let $(X,\mathcal{O}_X)$ be a smooth affine variety in $\mathbb{C}^N$, $Z\subset X$ a closed subvariety, and $Y:=X\setminus Z$. Then we have a long exact sequence:

$$\cdots\longrightarrow{}_{PG}H^q(X,Y;\mathcal{O}_X)\xrightarrow{j}{}_{PG}H^q(X;\mathcal{O}_X)\xrightarrow{i}{}_{PG}H^q(Y;\mathcal{O}_X)\xrightarrow{\delta}{}_{PG}H^{q+1}(X,Y;\mathcal{O}_X)\longrightarrow\cdots$$

<u>COROLLARY (3.3.2)</u> Let X,Y be as above. Then we have:

(i) $0\longrightarrow{}_{PG}H^0(X,Y;\mathcal{O}_X)\longrightarrow{}_{PG}H^0(X;\mathcal{O}_X)\longrightarrow{}_{PG}H^0(Y;\mathcal{O}_X)\longrightarrow$
$\longrightarrow{}_{PG}H^1(X,Y;\mathcal{O}_X)\longrightarrow 0$ is exact.

(ii) ${}_{PG}H^q(Y;\mathcal{O}_X)={}_{PG}H^{q+1}(X,Y;\mathcal{O}_X)$ for $q\geq 1$.

3.4. Relative PG-cohomology with coefficients in PG-coherent sheaves.

(3.4.1) Let $(X,\mathcal{O}_X)$, Z, Y be as above. Let $(\hat{A}_\sigma(X),\hat{A}_\sigma(Y))$ be a relative PG-covering of (X,Y) and $\mathcal{F}$ be a PG-coherent $\mathcal{O}_X$-Module. Then we have a chain of PG-syzygies:

$$0 \longrightarrow \mathcal{O}_X^{m_p} \xrightarrow{k_p} \mathcal{O}_X^{m_{p-1}} \xrightarrow{k_{p-1}} \cdots\cdots\cdots \xrightarrow{k_2} \mathcal{O}_X^{m_1} \xrightarrow{k_1} \mathcal{O}_X^{m_0} \xrightarrow{\varepsilon} \mathcal{F} \longrightarrow 0.$$

where k_i are PG-morphisms. For any integer $q \geqslant 0$, we have:

$$\begin{array}{ccccccc}
0 \longrightarrow & C^q(\hat{A}_\sigma(X),\hat{A}_\sigma(Y);\mathcal{O}_X^{m_p}) & \to\cdots\to & C^q(\hat{A}_\sigma(X),\hat{A}_\sigma(Y);\mathcal{O}_X^{m_0}) & \xrightarrow{\varepsilon} & C^q(\hat{A}_\sigma(X),\hat{A}_\sigma(Y);\mathcal{F}) & \to 0 \\
& \bigcup & & \bigcup & & & \\
0 \longrightarrow & {}_{PG}C^q(\hat{A}_\sigma(X),\hat{A}_\sigma(Y);\mathcal{O}_X^{m_p}) & \to\cdots\to & {}_{PG}C^q(\hat{A}_\sigma(X),\hat{A}_\sigma(Y);\mathcal{O}_X^{m_0}) & & &
\end{array}$$

So we define:

$${}_{PG}C^q(\hat{A}_\sigma(X),\hat{A}_\sigma(Y);\mathcal{F}) := \text{Image of}\left({}_{PG}C^q(\hat{A}_\sigma(X),\hat{A}_\sigma(Y);\mathcal{O}_X^{m_0}) \longrightarrow C^q(\hat{A}_\sigma(X),\hat{A}_\sigma(Y);\mathcal{F})\right).$$

Then the coboundary operator δ induces a homomorphism

$$\delta : {}_{PG}C^q(\hat{A}_\sigma(X),\hat{A}_\sigma(Y);\mathcal{F}) \longrightarrow {}_{PG}C^{q+1}(\hat{A}_\sigma(X),\hat{A}_\sigma(Y);\mathcal{F}),$$

and it is clear that $\delta^2 = 0$. Thus we have a relative cochain complex

$$({}_{PG}C^\bullet(\hat{A}_\sigma(X),\hat{A}_\sigma(Y);\mathcal{F}) , \delta) \text{ and define:}$$

$${}_{PG}H^q(\hat{A}_\sigma(X),\hat{A}_\sigma(Y);\mathcal{F}) := H^q({}_{PG}C^\bullet(\hat{A}_\sigma(X),\hat{A}_\sigma(Y);\mathcal{F})).$$

(3.4.2) Let $\tau : (\hat{A}_\rho(X),\hat{A}_\rho(Y)) \longrightarrow (\hat{A}_\sigma(X),\hat{A}_\sigma(Y))$ be a simplicial map associated to a refinement map $\tau : J \longrightarrow I$. Then induces a morphism

$$\tau^* : {}_{PG}C^q(\hat{A}_\sigma(X),\hat{A}_\sigma(Y);\mathcal{F}) \longrightarrow {}_{PG}C^q(\hat{A}_\rho(X),\hat{A}_\rho(Y);\mathcal{F})$$

and commutes with the coboundary operator δ. Consequently we have a homomorphism ${}_{PG}H^q(\hat{A}_\sigma(X),\hat{A}_\sigma(Y);\mathcal{F}) \longrightarrow {}_{PG}H^q(\hat{A}_\rho(X),\hat{A}_\rho(Y);\mathcal{F})$ which is independent of the choice of the refinement map τ.

(3.4.3) Similar to the case of the structure sheaf $\mathcal{O}_X$, we can also define;

$${}_{PG}H^q(X,Y;\mathcal{F}) = {}_{PG}H^q(X \bmod Y;\mathcal{F}) := \varinjlim_{\sigma} {}_{PG}H^q(\hat{A}_\sigma(X),\hat{A}_\sigma(Y);\mathcal{F}).$$

3.5. Long exact sequence.

PROPOSITION (3.5.1) Let $(X,\mathcal{O}_X)$ be a smooth affine variety in $\mathbb{C}^N$, $Z \subset X$ a closed subvariety and $Y := X \setminus Z$. Let $\mathcal{F}$ be a PG-coherent $\mathcal{O}_X$-Module. Then we have a long exact sequence:

$$\cdots \longrightarrow {}_{PG}H^q(X,Y;\mathcal{F}) \xrightarrow{j} {}_{PG}H^q(X;\mathcal{F}) \xrightarrow{i} {}_{PG}H^q(Y;\mathcal{F}) \xrightarrow{\delta} {}_{PG}H^{q+1}(X,Y;\mathcal{F}) \longrightarrow \cdots$$

COROLLARY (3.5.2) Let $X, Y\ \mathcal{F}$ be as above. Then we have:

(i) $0 \longrightarrow {}_{PG}H^0(X,Y;\mathcal{F}) \longrightarrow {}_{PG}H^0(X;\mathcal{F}) \longrightarrow {}_{PG}H^0(Y;\mathcal{F}) \xrightarrow{\delta}$
$\longrightarrow {}_{PG}H^1(X,Y;\mathcal{F}) \longrightarrow 0$ is exact.

(ii) ${}_{PG}H^q(Y;\mathcal{F}) = {}_{PG}H^{q+1}(X,Y;\mathcal{F})$ for $q \geqslant 1$.

§4. PG-cohomology with supports.

4.1. Supports.

(4.1.1) Let $(X,\mathcal{O}_X)$ be a smooth affine variety in $\mathbb{C}^N$, $Z \subset X$ a closed subvariety and $\hat{A}_\sigma(X)$ a PG-covering of X. If $\mathcal{F}$ is a PG-coherent $\mathcal{O}_X$-Module, then we have a cochain complex $({}_{PG}C^\bullet(\hat{A}_\sigma(X);\mathcal{F}), \delta)$. For a cochain $\varphi = (\varphi_{i_0\cdots i_q}) \in {}_{PG}C^q(A_\sigma(X);\mathcal{F})$, we define:

$$\mathrm{Supp}_{\hat{A}_\sigma(X)}(\varphi) := \overline{\bigcup_{\varphi_{i_0\cdots i_q}\neq 0} U_{i_0\cdots i_q}}$$

(4.1.2) Let Φ be a family of supports on X. Then we define:

$${}_{PG}C^q_\Phi(\hat{A}_\sigma(X);\mathcal{F}) := \left\{ \varphi \in {}_{PG}C^q(\hat{A}_\sigma(X);\mathcal{F}) \;\middle|\; \mathrm{Supp}_{\hat{A}_\sigma(X)}(\varphi) \in \Phi \right\}.$$

Since $\mathrm{Supp}_{\hat{A}_\sigma(X)}(\delta\varphi) \subset \mathrm{Supp}_{\hat{A}_\sigma(X)}(\varphi)$, we have

$$\begin{tikzcd}
{}_{PG}C^q(\hat{A}_\sigma(X);\mathcal{F}) \arrow[r, "\delta"] & {}_{PG}C^{q+1}(\hat{A}_\sigma(X);\mathcal{F}) \\
{}_{PG}C^q_\Phi(\hat{A}_\sigma(X);\mathcal{F}) \arrow[r, "\delta"] \arrow[u, hook] \arrow[ur, phantom, "\circlearrowleft"] & {}_{PG}C^{q+1}_\Phi(\hat{A}_\sigma(X);\mathcal{F}) \arrow[u, hook]
\end{tikzcd}$$

So we get a cochain complex (${}_{PG}C^{\bullet}_{\Phi}(\hat{A}_{\sigma}(X);\mathcal{F}),\delta$) and we define :

$$ {}_{PG}H^{q}_{\Phi}(\hat{A}_{\sigma}(X);\mathcal{F}) := H^{q}({}_{PG}C^{\bullet}_{\Phi}(\hat{A}_{\sigma}(X);\mathcal{F})). $$

4.2. PG-cohomology with supports.

(4.2.1) Let $\hat{A}_{\sigma}(X)=(U_i)_{i\in I}$ and $\hat{A}\rho(X)=(V_j)_{j\in J}$ be two PG-coverings of X such that $\hat{A}\sigma(X) < \hat{A}\rho(X)$. Then we have a refinement map $\tau: J \to I$ and τ induces a cochain map

$$ \tau^{*}: {}_{PG}C^{\bullet}(\hat{A}\sigma(X);\mathcal{F}) \longrightarrow {}_{PG}C^{\bullet}(\hat{A}\rho(X);\mathcal{F}). $$

It is easily seen that if $\mathrm{Supp}_{\hat{A}\sigma(X)}(\varphi)\in\Phi$, then $\mathrm{Supp}_{\hat{A}\rho(X)}(\tau^{*}\varphi)\in\Phi$.

So we have a cochain map

$$ \tau^{*}: {}_{PG}C^{\bullet}_{\Phi}(\hat{A}\sigma(X);\mathcal{F}) \longrightarrow {}_{PG}C^{\bullet}_{\Phi}(\hat{A}\rho(X);\mathcal{F}) $$

and consequently a homomorphism

$$ {}_{PG}H^{q}_{\Phi}(\hat{A}\sigma(X);\mathcal{F}) \longrightarrow {}_{PG}H^{q}_{\Phi}(\hat{A}\rho(X);\mathcal{F}) $$

which is independent of the choice of the refinement map τ .

(4.2.2) Thus we can define:

$$ {}_{PG}H^{q}_{\Phi}(X;\mathcal{F}) := \varinjlim_{\sigma} {}_{PG}H^{q}_{\Phi}(\hat{A}\sigma(X);\mathcal{F}). $$

(4.2.3) Similarly, for a closed subvariety $Z\subset X$, we define:

$$ {}_{PG}C^{q}_{Z}(\hat{A}\sigma(X);\mathcal{F}) := \left\{ \varphi \in {}_{PG}C^{q}(\hat{A}\sigma(X);\mathcal{F}) \;\middle|\; \mathrm{Supp}_{\hat{A}\sigma(X)}(\varphi)\subset Z \right\}, $$

$$ {}_{PG}H^{q}_{Z}(\hat{A}\sigma(X);\mathcal{F}) := H^{q}({}_{PG}C^{\bullet}_{Z}(\hat{A}\sigma(X);\mathcal{F})). $$

Now we can also define:

$$ {}_{PG}H^{q}_{Z}(X,\mathcal{F}) := \varinjlim_{\sigma} {}_{PG}H^{q}_{Z}(\hat{A}\sigma(X);\mathcal{F}). $$

(4.2.4) Let Φ be a family of supports on a smooth affine variety $(X,\mathcal{O}_X)$ in $\mathbb{C}^N$ and $\mathcal{F}$ be a PG-coherent $\mathcal{O}_X$-Module. Then we have :

$$ {}_{PG}H^{q}_{\Phi}(X,\mathcal{F}) = \varinjlim_{Z\in\Phi} {}_{PG}H^{q}_{Z}(X,\mathcal{F}) $$

where Z means a family of closed sets of X contained in Z.

4.3. Relations between relative PG-cohomology and PG-cohomology with supports.

(4.3.1) We define:

$$_{PG}\Gamma_Z(X;\mathcal{O}_X) := \{\phi\in{}_{PG}\Gamma(X;\mathcal{O}_X) \mid \mathrm{Supp}(\phi)\subset Z\},$$
$$\Gamma_Z(X_{alg},\mathcal{O}_{X_{alg}}) := \{\phi\in\Gamma(X_{alg},\mathcal{O}_{X_{alg}}) \mid \mathrm{Supp}(\phi)\subset Z\}.$$

<u>LEMMA(4.3.2)</u> Let $(X,\mathcal{O}_X)$, Z be as above. Then we have

$$_{PG}H^0_Z(X,\mathcal{O}_X) \xrightarrow{\sim} \Gamma_Z(X_{alg},\mathcal{O}_{X_{alg}}) \xrightarrow{\sim} {}_{PG}\Gamma_Z(X,\mathcal{O}_X).$$

<u>LEMMA(4.3.3)</u> Let $(X,\mathcal{O}_X)$, Z be as above. Then we have

$$_{PG}H^0(\hat{A}_\sigma(X),\hat{A}_\sigma(Y);\mathcal{O}_X) \xrightarrow{\sim} \Gamma_Z(X_{alg},\mathcal{O}_{X_{alg}})$$

where $Y:=X\setminus Z$.

<u>LEMMA(4.3.4)</u> Let X,Z,Y be as above. Then we have

$$_{PG}H^p_Z(X,\mathcal{O}_X) \xrightarrow{\sim} {}_{PG}H^p(X\mathrm{mod}Y,\mathcal{O}_X).$$

<u>PROPOSITION(4.3.5)</u> Let X,Z,Y be as above. Let $\mathcal{F}$ be a PG-coherent $\mathcal{O}_X$-Module. Then we have :

$$_{PG}H^p_Z(X,\mathcal{F}) \xrightarrow{\sim} {}_{PG}H^p(X\mathrm{mod}Y,\mathcal{F}).$$

<u>COROLLARY(4.3.6)</u> Let X,Z,Y, $\mathcal{F}$ be as above. Then we have :

(i) $0\longrightarrow {}_{PG}H^0_Z(X,\mathcal{F})\longrightarrow {}_{PG}H^0(X,\mathcal{F})\longrightarrow {}_{PG}H^0(Y,\mathcal{F})\longrightarrow {}_{PG}H^1_Z(X,\mathcal{F})\longrightarrow 0$

is exact.

(ii) $\quad {}_{PG}H^{q+1}_Z(X,\mathcal{F}) \simeq {}_{PG}H^q(Y,\mathcal{F}) \qquad$ for $q\geqslant 1$.

§5. PG-homology.

5.1. FN-structure, DFN-structure and PG-precosheaves.

(5.1.1) Let ${}_{PG}\Gamma(U,\mathcal{O}_{\mathbb{C}^n})$ be a vector space of PG-holomorphic functions on an open set $U\subset\mathbb{C}^n$. Then we can construct on ${}_{PG}\Gamma(U,\mathcal{O}_{\mathbb{C}^n})$ a locally convex topology as follows. Let

$$K_0\Subset K_1\Subset K_2\Subset K_3\Subset\cdots\cdots\Subset K_m\Subset\cdots\cdots\longrightarrow U,\qquad \bigcup_{m\geq 0}K_m=U$$

be a sequence of compact sets in U. ($K_i\Subset K_{i+1}$ means $K_i=\overset{\circ}{K}_{i+1}$.) Note that for any compact set $K\subset U$, there exists $N\geq 0$ such that $K\subset K_N$. Let

$$p_{m,k}(f):=\operatorname*{Max}_{z\in K_m}(1+|z|)^{-k}|f(z)|\quad\text{for}\quad f\in{}_{PG}\Gamma(U,\mathcal{O}_{\mathbb{C}^n})\quad(m=0,1,\cdots\cdots).$$

Then we have a countable sequence of semi-norms $(p_{m,k})$ $(m=0,1,2,\cdots\cdot,$ $k=0,1,2,\cdots\cdot)$. It follows from the Weierstrass convergence theorem that with this semi-norms ${}_{PG}\Gamma(U,\mathcal{O}_{\mathbb{C}^n})$ becomes complete and hence a Fréchet space. In other words this locally convex topology is given by a fundamental system $V(I;\varepsilon)$ of nbds of $0\in{}_{PG}\Gamma(U,\mathcal{O}_{\mathbb{C}^n})$:

$$V(I;\varepsilon):=\{f\in{}_{PG}\Gamma(U,\mathcal{O}_{\mathbb{C}^n})\ :\ p_{m,k}(f)<\varepsilon\ ,(m,k)\in I\ \}$$

where I runs through the finite subsets of $\mathbb{N}\times\mathbb{N}$.

It is easily seen that B is bounded in ${}_{PG}\Gamma(U,\mathcal{O}_{\mathbb{C}^n})$ if and only if for any $f\in B$ $\operatorname*{Max}_{z\in K_m}(1+|Z|)^{-k}|f(z)|\leq M_{m,k}<+\infty$ for every $m,k\in\mathbb{N}$.

(5.1.2) The locally convex space ${}_{PG}\Gamma(U,\mathcal{O}_{\mathbb{C}^n})$ is nuclear. This locally convex topology is compatible with the one induced from $\Gamma(U,\mathcal{O}_{\mathbb{C}^n})$.

(5.1.3) Let $(X,\mathcal{O}_X)$ be a smooth affine variety in $\mathbb{C}^N$. Then ${}_{PG}\Gamma(U,\mathcal{O}_X)$ has a locally convex structure of Fréchet-Nuclear for any open set $U\subset X$. In addition ${}_{PG}\Gamma(U,\mathcal{O}_X^s)={}_{PG}\Gamma(U,\mathcal{O}_X)^s$ is a Fréchet-Nuclear space in the Cartesian product topology. Again a sequence of compact sets in U as in (5.1.1) defines semi-norms $p_{m,k}$ by

$p_{m,k}(f_1,f_2,\cdots,f_s) := p_{m,k}(f_1) + p_{m,k}(f_2) + \cdots\cdots + p_{m,k}(f_s)$
and the semi-norms $(p_{m,k})$ determine the locally convex topology of ${}_{PG}\Gamma(U,\mathcal{O}_X^s)$ of type FN.

(5.1.4) Let $\mathcal{F}$ be a PG-coherent $\mathcal{O}_X$-Module. Then for any open set $U \subset X$, the space $\Gamma(U,\mathcal{F})$ has a natural topological structure of Fréchet-Nuclear. If we have a chain of PG-syzygies :

$$0 \longrightarrow \mathcal{O}_X^{m_p} \xrightarrow{k_p} \mathcal{O}_X^{m_{p-1}} \xrightarrow{k_{p-1}} \cdots\cdots\cdots \longrightarrow \mathcal{O}_X^{m_1} \xrightarrow{k_1} \mathcal{O}_X^{m_0} \xrightarrow{\varepsilon} \mathcal{F} \longrightarrow 0$$

where k_i are PG-morphisms, then we set

$${}_{PG}\Gamma(U,\mathcal{F}) := \overline{\text{Image of } (\ {}_{PG}\Gamma(U,\mathcal{O}_X^{m_0}) \xrightarrow{\ \varepsilon\ } \Gamma(U,\mathcal{F})\ )}$$

Then ${}_{PG}\Gamma(U,\mathcal{F})$ is a closed subspace of $\Gamma(U,\mathcal{F})$ and has a FN-structure induced from $\Gamma(U,\mathcal{F})$.

(5.1.5) We define the space $\mathcal{DF}(U)$ as the strong dual of ${}_{PG}\Gamma(U,\mathcal{F})$. Then $\mathcal{DF}(U)$ has a topological vector space structure of type DFN.

(5.1.6) Then a PG-precosheaf $\mathcal{DF}$ is defined as follows:
For an open set $U \subset X$, $\mathcal{DF} : U \longmapsto \mathcal{DF}(U)$ is a functor. The extension map $\rho_U^V : \mathcal{DF}(V) \longrightarrow \mathcal{DF}(U)$ is defined as the transposed of the natural restriction map $\mathrm{res}_V^U : {}_{PG}\Gamma(U,\mathcal{F}) \longrightarrow {}_{PG}\Gamma(V,\mathcal{F})$ for an open set $V \subset U$.

5.2. Supports.

(5.2.1) Let $\hat{A}_\sigma(X) = (U_i)_{i\in I}$ be a PG-covering of X. Let $N(\hat{A}_\sigma(X))$ denote the nerve of the covering $\hat{A}_\sigma(X)$. For a element $g \in \prod_{i_0,\cdots i_p} \mathcal{DF}(U_{i_0\cdots i_p})$, we define the support as following:

$$\mathrm{Supp}_{N(\hat{A}_\sigma(X))}(g) := \{\ s=(i_0,\ldots,i_p) \in I^{p+1} \mid g_{i_0\cdots\cdots i_p} \neq 0\ \}.$$

(5.2.2) We define the subcomplex $K(g)$ of $N(\hat{A}_\sigma(X))$ as the set of all simplices σ, $\sigma<s$ for some $s\in \mathrm{Supp}_{N(\hat{A}_\sigma(X))}(g)$. Note that $K(g)$ is locally finite in $N(\hat{A}_\sigma(X))$, i.e., for every $s\in N(\hat{A}_\sigma(X))$ the set $\{t\in K(g) \mid (t,s)\in N(\hat{A}_\sigma(X))\}$ is finite, since $\hat{A}_\sigma(X)$ is an open covering by relatively compact open sets which is locally finite.

5.3. PG-homology with coefficients in PG-precosheaves.

(5.3.1) We define :

$$C_p(\hat{A}_\sigma(X), \mathcal{DF}) := \prod_{(i_0,\cdots,i_p)} \mathcal{DF}(U_{i_0\cdots i_p}).$$

(5.3.2) We can define the coboundary operator

$$\partial_{p-1} : C_p(\hat{A}_\sigma(X), \mathcal{DF}) \longrightarrow C_{p-1}(\hat{A}_\sigma(X), \mathcal{DF})$$

by the formula

$$\partial_{p-1}(g_{i_0\cdots\cdots i_p}) := \sum_{j=0}^{p} (-1)^j \rho^{U_{i_0\cdots\cdots i_p}}_{U_{i_0\cdots\hat{i}_j\cdots i_p}} (g_{i_0\cdots\cdots i_p})$$

where $\rho^{U_{i_0\cdots\cdots i_p}}_{U_{i_0\cdots\hat{i}_j\cdots i_p}}$ is the tranposed operater of the restriction map $\mathrm{res}^{U_{i_0\cdots\hat{i}_j\cdots i_p}}_{U_{i_0\cdots\cdots i_p}}$.

(5.3.3) Since $K(\partial_{p-1}(g)) \subset K(g)$, the boundary operator ∂_{p-1} is well-defined and we obtain thus a chain complex $(C_\bullet(\hat{A}_\sigma(X), \mathcal{DF}), \partial)$. Their p- th homology will denote by $H_p(\hat{A}_\sigma(X), \mathcal{DF})$.

5.4. PG-homology with supports.

(5.4.1) For $g\in C_p(\hat{A}_\sigma(X), \mathcal{DF})$, we define: $\mathrm{Supp}_{\hat{A}_\sigma(X)}(g) := \overline{\bigcup_{t\in K(g)} U_t}$.

(5.4.2) If Ψ is a family of supports on X, then we define :

$$C^\Psi_p(\hat{A}_\sigma(X), \mathcal{DF}) := \{ g\in C_p(\hat{A}_\sigma(X), \mathcal{DF}) \mid \mathrm{Supp}_{\hat{A}_\sigma(X)}(g) \in \Psi \}.$$

One readily verifies that $\mathrm{Supp}_{\hat{A}_\sigma(X)}(\partial_{p-1} g) \subset \mathrm{Supp}_{\hat{A}_\sigma(X)}(g)$. Therefore we have a subcomplex $(C^\Psi_\bullet(\hat{A}_\sigma(X), \mathcal{DF}), \partial)$ of $(C_\bullet(\hat{A}_\sigma(X), \mathcal{DF}), \partial)$.

The homology of above complex will be defined by

$$H_q^{\Psi}(\hat{A}_{\sigma}(X), \mathcal{DF}) := H_q(\, C_{\bullet}^{\Psi}(\hat{A}_{\sigma}(X), \mathcal{DF})\,).$$

(5.4.3) Let $\hat{A}_{\sigma}(X) = (U_i)_{i \in I}$ and $\hat{A}_{\rho}(X) = (V_j)_{j \in J}$ be two PG- coverings of X such that $\hat{A}_{\sigma}(X) < \hat{A}_{\rho}(X)$. Then a refinement map $\tau : J \rightarrow I$ induces a chain map $\tau_* : C_{\bullet}(\hat{A}_{\rho}(X), \mathcal{DF}) \longrightarrow C_{\bullet}(\hat{A}_{\sigma}(X), \mathcal{DF})$.

For $g \in C_{\bullet}(\hat{A}_{\rho}(X), \mathcal{DF})$, we can see $\mathrm{Supp}_A(\tau_*(g)) \subset \overline{\bigcup_{U_i \cap B \neq \emptyset} U_i}$, where $A = \hat{A}_{\sigma}(X)$, $B = \mathrm{Supp}_{\hat{A}_{\rho}(X)}(g)$. Hence ,if $\mathrm{Supp}_{\hat{A}_{\rho}(X)}(g) \in \Psi$, then $\mathrm{Supp}_{\hat{A}_{\sigma}(X)}(\tau_*(g)) \in \Psi$. So we have a chain map

$$\tau_* : C_{\bullet}^{\Psi}(\hat{A}_{\rho}(X), \mathcal{DF}) \longrightarrow C_{\bullet}^{\Psi}(\hat{A}_{\sigma}(X), \mathcal{DF}),$$

and consequently a homomorphism

$$H_p^{\Psi}(\hat{A}_{\rho}(X), \mathcal{DF}) \longrightarrow H_p^{\Psi}(\hat{A}_{\sigma}(X), \mathcal{DF}),$$

which is independent of the choice of the refinement map τ .

(5.4.4) Thus we can define :

$$H_p^{\Psi}(X, \mathcal{DF}) := \varprojlim_{\sigma} H_p^{\Psi}(\hat{A}_{\sigma}(X), \mathcal{DF}).$$

(5.4.5) Similarly, for a closed subvariety $Z \subset X$, we define :

$$C_p^Z(\hat{A}_{\sigma}(X); \mathcal{DF}) := \{\, g \in C_p(\hat{A}_{\sigma}(X); \mathcal{DF}) \ :\ \mathrm{Supp}_{\hat{A}_{\sigma}(X)}(g) \subset Z \,\}.$$

Then we get a subcomplex $(\, C_{\bullet}^Z(\hat{A}_{\sigma}(X); \mathcal{DF}), \partial\,)$ and define :

$$H_p^Z(\hat{A}_{\sigma}(X); \mathcal{DF}) := H_p(\, C_{\bullet}^Z(\hat{A}_{\sigma}(X); \mathcal{DF})\,).$$

Now we can also define :

$$H_p^Z(X; \mathcal{DF}) := \varprojlim_{\sigma} H_p^Z(\hat{A}_{\sigma}(X); \mathcal{DF}).$$

§6. Relative duality.

6.1. Dual pair of supports.

(6.1.1) Let Φ be a family of supports on a topological space X. The dual family Ψ of Φ is defined as follows :

$$\Psi := \{ T \subset X \text{ is closed} \mid T \cap S \text{ is compact for all } S \in \Phi \} \quad .$$

The dual family Ψ is also a family of suppots on X.

(6.1.2) A pair of families of supports (Φ,Ψ) is called dual pair of families of supports if

(i) Φ is the dual family of Ψ , and

(ii) Ψ is the dual family of Φ .

If X is locally compact and (Φ,Ψ) is a dual pair of familles of supports on X, then $\bigcup_{S\in\Phi} S = X$ and $\bigcup_{Z\in\Psi} Z = X$.

6.2. Locally convex topology on PG-chains and PG-cochains.

(6.2.1) Let Ψ be a family of supports on a smooth affine variety $(X,\mathcal{O}_X)$ in $\mathbb{C}^N$ and $\hat{A}_\sigma(X)=(U_i)_{i\in I}$ be a PG-covering of X. For any set $Z\subset X$, we define :

$$\hat{A}_\sigma(X)|_Z := \{ U_i \in \hat{A}_\sigma(X) \quad : \quad U_i \subset Z \}$$

and if $Z\in\Psi$, then we now define :

$$C_q(\hat{A}_\sigma(X)|_Z ; \mathcal{DF}) := \prod_{U_{i_0},\dots,U_{i_q}\in\hat{A}_\sigma(X)|_Z} \mathcal{DF}(U_{i_0\dots i_q}) \quad .$$

This is a countable product of locally convex spaces of type DFN. Hence we get a subcomplex ($C_\bullet(\hat{A}_\sigma(X)|_Z;\mathcal{DF})$, ∂) of PG-chain complex ($C^\Psi_\bullet(\hat{A}_\sigma(X);\mathcal{DF})$, ∂).

(6.2.2) Let $Z,Z'\in\Psi$ such that $Z\subset Z'$. Then the chain map $N(\hat{A}_\sigma(X)|_Z) \longrightarrow N(\hat{A}_\sigma(X)|_{Z'})$ induces a chain map :

$$j^Z_{Z'} : C_q(\hat{A}_\sigma(X)|_Z;\mathcal{DF}) \longrightarrow C_q(\hat{A}_\sigma(X)|_{Z'};\mathcal{DF})$$

given by

$$j^Z_{Z'}(g)_{i_0\cdots i_q} := \begin{cases} g_{i_0\cdots i_q} & \text{if } U_{i_0},\dots,U_{i_q} \in \hat{A}_\sigma(X)|_Z \\ 0 & \text{otherwise} \end{cases}$$

where $g=(g_{i_0\cdots i_q}) \in C_q(\hat{A}_\sigma(X)|_Z;\mathcal{DF})$.

(6.2.3) Hence we get a system of injective topological homomorphisms on the partially ordered set Ψ (ordered by inclsion). Obviously

$$C^\Psi_q(\hat{A}_\sigma(X);\mathcal{DF}) = \bigcup_{Z\in\Psi} C_q(\hat{A}_\sigma(X)|_Z;\mathcal{DF}) .$$

So we can define a locally convex topology on $C^\Psi_q(\hat{A}_\sigma(X);\mathcal{DF})$ by setting:
$$C^\Psi_q(\hat{A}_\sigma(X);\mathcal{DF}) := \varinjlim_{Z\in\Psi} C_q(\hat{A}_\sigma(X)|_Z;\mathcal{DF}).$$

(6.2.4) Let (Φ,Ψ) be a dual pair of families of supports on X. Then for $S\in\Phi$, we have a exact sequence

$${}_{PG}C^q(\hat{A}_\sigma(X);\mathcal{F}) \xrightarrow{\text{proj}} {}_{PG}C^q(\hat{A}_\sigma(X)|_{X\setminus S};\mathcal{F}) \longrightarrow 0$$

where proj is the projection. We define :

$${}_{PG}C^q({}_S|\hat{A}_\sigma(X);\mathcal{F}) := \mathrm{Ker}(\,{}_{PG}C^q(\hat{A}_\sigma(X);\mathcal{F}) \longrightarrow {}_{PG}C^q(\hat{A}_\sigma(X)|_{X\setminus S};\mathcal{F}))$$

Then we get a subcomplex $(\,{}_{PG}C^\bullet({}_S|\hat{A}_\sigma(X);\mathcal{F}),\ \delta\,)$.

(6.2.5) For any two $S,S'\in\Phi$ such that $S\subset S'$, the projection

$${}_{PG}C^q(\hat{A}_\sigma(X)|_{X\setminus S};\mathcal{F}) \longrightarrow {}_{PG}C^q(\hat{A}_\sigma(X)|_{X\setminus S'};\mathcal{F})$$

induces the natural map:

$${}_{PG}C^q({}_S|\hat{A}_\sigma(X);\mathcal{F}) \longrightarrow {}_{PG}C^q({}_{S'}|\hat{A}_\sigma(X);\mathcal{F}).$$

Hence we have

$${}_{PG}C^q_\Phi(\hat{A}_\sigma(X);\mathcal{F}) = \varinjlim_{S\in\Phi} {}_{PG}C^q({}_S|\hat{A}_\sigma(X);\mathcal{F})$$

(6.2.6) On the other hand, we have natural topological inclusions:

$${}_{PG}C^q(\hat{A}_\sigma(X)|_S;\mathcal{F}) \hookrightarrow {}_{PG}C^q({}_S|\hat{A}_\sigma(X);\mathcal{F}) \hookrightarrow {}_{PG}C^q(\hat{A}_\sigma(X)|_{S'};\mathcal{F}).$$

So, defining the extension maps :

$$j^S_{S'} : {}_{PG}C^q(\hat{A}_\sigma(X)|_S;\mathcal{F}) \longrightarrow {}_{PG}C^q(\hat{A}_\sigma(X)|_{S'};\mathcal{F})$$

by

$$j^S_{S'}(f)_{i_0\cdots\cdots i_q} := \begin{cases} f_{i_0\cdots\cdots i_q} & \text{if} \quad U_{i_0\cdots\cdots i_q} \subset S \\ 0 & \text{otherwise.} \end{cases}$$

We can also write the above locally convex topology as :

$${}_{PG}C^q_\Phi(\hat{A}_\sigma(X);\mathcal{F}) = \varinjlim_{S\in\Phi} {}_{PG}C^q(\hat{A}_\sigma(X)|_S;\mathcal{F}).$$

6.3. Natural pairing between chains and cochains.

(6.3.1) Let $(X,\mathcal{O}_X)$ be a smooth affine variety in $\mathbb{C}^N$ and $\mathcal{F}$ a PG-coherent $\mathcal{O}_X$-Module. For any open set $U\subset X$, we have a natural pairing :

$$\mathcal{F}(U) \times \mathcal{D}\mathcal{F}(U) \longrightarrow \mathbb{C}$$

given by

$$(s, f) \longmapsto \langle s, f\rangle = f(s).$$

(6.3.2) Let $\hat{A}_\sigma(X)$ be a PG-covering of X and $(\Phi;\Psi)$ be a dual pair of families of supports. Then we have a natural pairing :

$${}_{PG}C^q_\Phi(\hat{A}_\sigma(X);\mathcal{F}) \times C^\Psi_q(\hat{A}_\sigma(X);\mathcal{D}\mathcal{F}) \longrightarrow \mathbb{C}$$

defined by

$$(f,g) \longmapsto \langle f,g\rangle = \sum_{(i_0,\cdots,i_q)} \langle f_{i_0\cdots\cdots i_q}, g_{i_0\cdots\cdots i_q}\rangle$$

where $f=(f_{i_0\cdots\cdots i_q})\in{}_{PG}C^q_\Phi(\hat{A}_\sigma(X);\mathcal{F})$ and $g=(g_{i_0\cdots i_q})\in C^\Psi_q(\hat{A}_\sigma(X);\mathcal{D}\mathcal{F})$. This pairing is meaningful because of the finiteness of the sum and it is compatible with the boundary operator and coboundary operator.

(6.3.3) The above natural pairing is separated,i.e.,

$$(\forall f\in{}_{PG}C^q_\Phi(\hat{A}_\sigma(X);\mathcal{F})\setminus\{0\})(\exists g\in C^\Psi_q(\hat{A}_\sigma(X);\mathcal{D}\mathcal{F})): \quad \langle f,g\rangle \neq 0$$

$$(\forall g\in C^\Psi_q(\hat{A}_\sigma(X);\mathcal{D}\mathcal{F})\setminus\{0\})(\exists f\in{}_{PG}C^q_\Phi(\hat{A}_\sigma(X);\mathcal{F})): \quad \langle f,g\rangle \neq 0.$$

Theorem (6.3.4) Let $(X, \mathcal{O}_X)$ be a smooth affine variety in $\mathbb{C}^N$ and (Φ, Ψ) be a dual pair of families of supports on X. Then for any PG-coherent $\mathcal{O}_X$-Module $\mathcal{F}$, we have topological isomorphisms :

$$H_q^{\Psi}(\hat{A}_\sigma(X), \mathcal{D}\mathcal{F}) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{cont}}({}_{PG}H_{\Phi}^{q}(\hat{A}_\sigma(X), \mathcal{F}); \mathbb{C})$$

$${}_{PG}H_{\Phi}^{q}(\hat{A}_\sigma(X), \mathcal{F}) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{cont}}(H_q^{\Psi}(\hat{A}_\sigma(X), \mathcal{D}\mathcal{F}); \mathbb{C})$$

for any $q \geqslant 0$, where $\mathrm{Hom}_{\mathrm{cont}}$ denotes the weak dual.